Proof-Theoretic Soundness and Completeness

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Abstract

We give a calculus for reasoning about the first-order fragment of classical logic that is adequate for giving the truth conditions of intuitionistic Kripke frames, and outline a proof-theoretic soundness and completeness proof, which we believe is conducive to automation.

1 A Semantic Calculus for Intuitionistic Kripke Models

In Rothenberg (2010), we use correspondence theory (Blackburn et al., 2001) to give a cut-free calculus for reasoning about intuitionistic Kripke models (Kripke, 1965) using a fragment of first-order classical logic.

Definition 1 (Partially-Shielded Formulae)**.** We define the partially-shielded fragment (PSF) of first-order formulae: (1) $\bot$; (2) $P\{x\}$ iff P is an atomic propositional variable, or an atomic first-order formula with a free variable x; (3) $A\{x\} \wedge B\{x\}$ and $A\{x\} \vee B\{x\}$, iff $A\{x\}, B\{x\}$ are in PSF; (4) $\mathcal{R}xy$, where $\mathcal{R}$ is a *fixed* atomic binary relation (5) $\forall y.(\mathcal{R}xy \wedge A\{y\}) \rightarrow B\{y\}$, iff $A\{x\}$ and $B\{x\}$ are in PSF.

Proposition 1. *A formula in* PSF *is either of the form* $\mathcal{R}xy$ *or has at most one free variable.*

Proof. By induction on the structure of the formula. □

We give the calculus $\mathbf{G3c/PSF}$ in Figure 1, which is useful for reasoning about sequents of formulae in PSF. A variant of it was introduced in Rothenberg (2010), based on ideas from a calculus for the guarded fragment (GF) of first-order formulae given in Dyckhoff and Simpson (2006).

$$\frac{}{\Gamma, P \Rightarrow P, \Delta}\ \mathsf{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \mathsf{L}\bot$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ \mathsf{R}\wedge \qquad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \mathsf{R}\vee$$

$$\frac{\Gamma, \mathcal{R}xz, \forall y. \ldots \Rightarrow A\{z\}, \Delta \quad \Gamma, \mathcal{R}xz, \forall y. \ldots, B\{z\} \Rightarrow \Delta}{\Gamma, \mathcal{R}xz, \forall y.(\mathcal{R}xy \wedge A\{y\}) \rightarrow B\{y\} \Rightarrow \Delta}\ \mathsf{L}\forall\rightarrow \qquad \frac{\Gamma, \mathcal{R}xz, A\{z\} \Rightarrow B\{z\}, \Delta}{\Gamma \Rightarrow \forall y.(\mathcal{R}xy \wedge A\{y\}) \rightarrow B\{y\}, \Delta}\ \mathsf{R}\forall\rightarrow$$

Figure 1: The calculus $\mathbf{G3c/PSF}$ for sequents of partially shielded formulae.

In Figure 1, the variable y is fresh for the conclusion of the $\mathsf{R}\forall\rightarrow$ rule, and that $\forall y. \ldots$ in the premisses of the $\mathsf{L}\forall\rightarrow$ and $\mathsf{R}\forall\rightarrow$ rules is an abbreviation of "$\forall y.(\mathcal{R}xy \wedge A\{y\}) \rightarrow B\{y\}$".

Proposition 2 (Standard Structural Rules, Rothenberg (2010))**.** *The following rules are admissible in* $\mathbf{G3c/PSF}$*:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ \mathsf{W} \qquad \frac{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta', \Delta', \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ \mathsf{C} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ \mathsf{Cut}$$

Proposition 3 (Negri (2007))**.** *Let* $\mathbf{G3c/PSF}^*$ *be* $\mathbf{G3c/PSF}$ *plus the following (geometric) rules:*

$$\frac{\mathcal{R}xx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{refl} \qquad \frac{\mathcal{R}xz, \mathcal{R}xy, \mathcal{R}yz, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, \mathcal{R}yz, \Gamma \Rightarrow \Delta}\ \mathsf{tran} \qquad \frac{\mathcal{R}xy, Px, Py, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, Px, \Gamma \Rightarrow \Delta}\ \mathsf{mono}$$

where Px, Py *in the* mono *rule are atomic.*

Corollary 4 (Negri (2007)). *The standard structural rules (Proposition 2) are admissible in* $\mathbf{G3c}/\mathbf{PSF}^*$.

Remark 1. Earlier work on geometric rules for modal logics can be found in Simpson (1994).

Remark 2. The labelled sequent calculus $\mathbf{G3I}$ (Negri, 2007) and (Dyckhoff and Negri, 2011) can be thought of as an alternative form of $\mathbf{G3c}/\mathbf{PSF}^*$ that hides the quantifiers and incorporates the mono rule into the axiom Ax.

Definition 2 (Translation of Propositional Formulae into PSF).

$$\begin{array}{lll} \bot^\dagger =_{def} \bot & (A \wedge B)^\dagger =_{def} A^\dagger \wedge B^\dagger & (A \to B)^\dagger =_{def} \forall y.(\mathcal{R}xy \wedge A^\dagger) \to B^\dagger \\ P^\dagger =_{def} \hat{P}x & (A \vee B)^\dagger =_{def} A^\dagger \vee B^\dagger & \end{array}$$

where the translation of $A \to B$ requires that the free variable of $A^\dagger, B^\dagger$ is x, and $y \neq x$, and $\hat{P}x$ uniquely corresponds to P. Recall that $\mathcal{R}$-formulae occur only as strict subformulae in the translation. The extension is adapted to sequents naturally, where all formulae have the same free variable.

Definition 3 (Kripke Semantics of PSF). Let $\mathfrak{M} = \langle W, R, \Vdash \rangle$ be a Kripke model, and let $\hat{x}$ be a function from first-order variables into W. Then

1. $\mathfrak{M} \nVdash \bot$ iff $\mathfrak{M}, \hat{x} \nVdash \bot$ for all $\hat{x} \in W$;
2. $\mathfrak{M} \Vdash P\{x\}$ iff
   (a) $\mathfrak{M} \Vdash P\{x\}$ iff $\mathfrak{M}, \hat{x} \Vdash P\{x\}$ for all $\hat{x} \in W$, where $P\{x\}$ is an atomic propositional variable;
   (b) $\mathfrak{M} \Vdash P\{x\}$ iff $\mathfrak{M}, \hat{x} \Vdash Px$ for some $\hat{x} \in W$, where $P\{x\}$ is an atomic first-order formula;
3. $\mathfrak{M} \Vdash A \wedge B$ iff $\mathfrak{M} \Vdash A$ and $\mathfrak{M} \Vdash B$;
4. $\mathfrak{M} \Vdash A \vee B$ iff either $\mathfrak{M} \Vdash A$ or $\mathfrak{M} \Vdash B$;
5. $\mathfrak{M} \Vdash \mathcal{R}xy$ iff $(\hat{x}, \hat{y}) \in R$;
6. $\mathfrak{M} \Vdash (\mathcal{R}xy \wedge A\{y\}) \to B\{y\}$ iff $\mathfrak{M}, \hat{x} \Vdash (\mathcal{R}xy \wedge A\{y\}) \to B\{y\}$ iff $\mathfrak{M} \Vdash \mathcal{R}xy$ and either $\mathfrak{M} \nVdash A\{y\}$ or $\mathfrak{M} \Vdash B\{y\}$.

This is extended naturally for sequents of formulae by $\mathfrak{M} \Vdash \Gamma \Rightarrow \Delta$ iff either $\mathfrak{M} \nVdash \bigwedge\Gamma$ or $\mathfrak{M} \Vdash \bigvee\Delta$.

Theorem 5 (Soundness and Completeness, Rothenberg (2010)). *Let* $\mathfrak{M} = \langle W, R, \Vdash \rangle$ *be a Kripke model for* $\mathbf{Int}$. *Then* $\mathfrak{M} \vDash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{G3c}/\mathbf{PSF}^* \vdash \Gamma^\dagger \Rightarrow \Delta^\dagger$.

Proof. Using Definition 3, we note the rules of $\mathbf{G3c}/\mathbf{PSF}^*$ are sound w.r.t. the properties of $\mathfrak{M}$. For completeness, we show by induction of the structure of sequents (the sizes of Γ, Δ and the structure of each formula). □

Lemma 6 (Right Monotonicity). *The rule*

$$\frac{\mathcal{R}xy, \Gamma \Rightarrow \Delta, Px, Py}{\mathcal{R}xy, \Gamma \Rightarrow \Delta, Py}\ \mathsf{mono}'$$

is admissible in $\mathbf{G3c}/\mathbf{PSF}^*$.

Proof. Using cut. □

Lemma 7 (General Monotonicity). *The rules*

$$\frac{\mathcal{R}xy, Ax, Ay, \Gamma \Rightarrow \Delta}{\mathcal{R}xy, Ax, \Gamma \Rightarrow \Delta} \qquad \frac{\mathcal{R}xy, \Gamma \Rightarrow \Delta, Ax, Ay}{\mathcal{R}xy, \Gamma \Rightarrow \Delta, Ay}$$

are admissible in $\mathbf{G3c}/\mathbf{PSF}^*$.

Proof. By induction on the derivation depth and formula size. □

Theorem 8. *Let* $\mathbf{G}$ *be a multisuccedent sequent calculus for* $\mathbf{Int}$*, e.g.* $\mathbf{m\text{-}G3ip}$ *(Troelstra and Schwichtenberg, 2000). Then* $\mathbf{G} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{G3c/PSF}^* \vdash \Gamma^\dagger \Rightarrow \Delta^\dagger$.

Proof. By induction on the derivation height. An outline of the proof is as follows: (1) Hyperextend (Avron, 1991) $\mathbf{G}$ to a hypersequent calculus $\mathbf{HG}$; (2) Show $\mathbf{G} \vdash \Gamma \Rightarrow \Delta$ iff $\mathbf{HG} \vdash \Gamma \Rightarrow \Delta \mid \mathcal{H}$ (straightforward). (3) Extend Definition 2 so that components in hypersequents are translated with unique free variables; (4) Show $\mathbf{HG} \vdash \mathcal{H}$ iff $\mathbf{G3c/PSF}^* \vdash \mathcal{H}^\dagger$. (Note that instances of mono or trans can be eliminated from $\mathbf{G3c/PSF}^*$ proofs of sequents with a single free variable.) □

Corollary 9 (Soundness and Completeness)**.** *Let* $\mathbf{G}$ *be a multisuccedent sequent calculus for* $\mathbf{Int}$*. Then* $\mathbf{G}$ *is sound and complete w.r.t.* $\mathbf{Int}$.

Proof. Follows from Theorem 8. □

2 Future Work

We expect that adapting this work to single-succedent calculi, e.g. $\mathbf{G3ip}$ (Troelstra and Schwichtenberg, 2000), should be straightforward. An obvious extension is to adapt this work to hypersequent calculi for superintuitionistic logics, e.g. in (Avron, 1991). For logics with geometric Kripke semantics (Rothenberg, 2010), this should be straightforward. This work can be adapted to cut-free sequent calculi for modal logics in a straightforward manner, using similar calculi for guarded formulae, such as Dyckhoff and Simpson (2006). The reader is also referred to Ohlbach et al. (2001) for related work.

Adapting this work to extensions of Gentzen calculi should be possible, by applying translations of their data structures into sequents of PSF, and using a limited form of proof search on schematic rules. (Such work may be easier, if the data structure allows relations between points in a Kripke frame to be explicit.)

Theorem 8 may be extended to $\mathbf{HG} \vdash (\Gamma \Rightarrow \Delta)^\bullet$ iff $\mathbf{G3c/PSF} \vdash \Gamma \Rightarrow \Delta$, where $(\Gamma \Rightarrow \Delta)^\bullet$ is a translation from sequents of PSF into hypersequents, based on the "transitive unfolding" procedure from (Rothenberg, 2010).

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